Brownian motion and Stochastic Calculus
Dylan Possamaï

Assignment 7—solutions

Exercise 1

Let $(B_t)_{t\in[0,1]}$ be a Brownian motion on $(\Omega,\mathcal{F},\mathbb{P})$ and define the process $(M_t)_{t\geq 0}$ by $M_t := \sup_{0\leq s\leq t} B_s$. Consider the random variable

$$D := \sup_{0\leq s\leq 1}\bigg\{\sup_{0\leq t\leq s}\{B_t - B_s\}\bigg\}.$$

That is, D characterises the maximal possible 'downfall' in trajectories of the Brownian motion on the time interval $[0,1]$.

1) Show that $D \overset{\text{law}}{=} \sup_{0\leq t\leq 1}|B_t|$.

   Hint: you can use (and prove if you want!) Lévy's theorem, which states that the processes $M-B$ and $|B|$ have the saw law under $\mathbb{P}$.

2) Show that $\sup_{0\leq t\leq 1}|B_t| \overset{\text{law}}{=} 1/\sqrt{\bar{T}_1}$, where $\bar{T}_1 := \inf\{t>0 : |B_t|\geq 1\}$.

3) Conclude that $\mathbb{E}^{\mathbb{P}}[D] = \sqrt{\pi/2}$.

1) **Let $Z_t := M_t - B_t$ and $Y_t := |B_t|$. With the definition of D we have to check that**

$$\sup_{0\leq t\leq 1} Z_t \overset{\text{law}}{=} \sup_{0\leq t\leq 1} Y_t.$$

Since both Z and Y are continuous processes, it suffices to check that

$$\sup_{t\in[0,1]\cap\mathbb{Q}} Z_t \overset{\text{law}}{=} \sup_{t\in[0,1]\cap\mathbb{Q}} Y_t. \tag{0.1}$$

Let $(t_n)_{n\in\mathbb{N}}$ be a counting sequence in $[0,1]\cap\mathbb{Q}$. By Lévy's theorem, the processes Z and Y have the same law, and therefore for $n\in\mathbb{N}$ the random variables

$$Z_n := \sup(Z_{t_1}, Z_{t_2},\dots,Z_{t_n}),\ Y_n := \sup(Y_{t_1},Y_{t_2},\dots,Y_{t_n}),$$

have the same law. Since Z_n and Y_n converge monotonically to $\sup_{t\in[0,1]\cap\mathbb{Q}} Z_t$ and $\sup_{t\in[0,1]\cap\mathbb{Q}} Y_t$ we have for all $x\in\mathbb{R}$

$$\begin{aligned}
\mathbb{P}\bigg[\sup_{t\in[0,1]\cap\mathbb{Q}} Z_t\leq x\bigg] &= \mathbb{P}\bigg[\bigcap_{n=0}^{+\infty}\{Z_n\leq x\}\bigg]\\
&= \lim_{n\to+\infty}\mathbb{P}[Z_n\leq x]\\
&= \lim_{n\to+\infty}\mathbb{P}[Y_n\leq x]\\
&= \mathbb{P}\bigg[\sup_{t\in[0,1]\cap\mathbb{Q}} Y_t\leq x\bigg],
\end{aligned}$$

which yields (0.1).

2) **We recall the self-similarity property of Brownian motion, *i.e.*, for $c>0$**

$$(cB_{t/c^2})_{t\geq 0} \overset{\text{law}}{=} (B_t)_{t\geq 0}.$$

Therefore, for $x > 0$

$$\begin{aligned}\mathbb{P}\left[\sup_{0\leq t\leq 1}|B_t|\leq x\right] &= \mathbb{P}[\sup_{0\leq t\leq 1}|B_{t/x^2}|\leq 1]\\ &= \mathbb{P}\left[\sup_{0\leq t\leq 1/x^2}|B_t|\leq 1\right]\\ &= \mathbb{P}[\bar{T}_1\geq x^{-2}]\\ &= \mathbb{P}[1/\sqrt{\bar{T}_1}\leq x].\end{aligned}$$

3) **Using the identity**

$$\sqrt{2/\pi}\int_0^\infty \mathrm{e}^{-x^2/(2\sigma^2)}\mathrm{d}x=\sigma,$$

and Tonelli's theorem we have

$$\mathbb{E}^{\mathbb{P}}[D]=\mathbb{E}^{\mathbb{P}}\left[\sup_{0\leq t\leq 1}|B_t|\right]=\mathbb{E}^{\mathbb{P}}[1/\sqrt{\bar{T}_1}]=\sqrt{2/\pi}\int_0^\infty \mathbb{E}^{\mathbb{P}}[\mathrm{e}^{-x^2\bar{T}_1/2}]\mathrm{d}x.$$

From a previous exercise, we know that the Laplace transform of $\bar{T}_1$ is

$$\mathbb{E}^{\mathbb{P}}[\mathrm{e}^{-\mu\bar{T}_1}]=1/\cosh(\sqrt{2\mu}),\ \forall\mu>0.$$

Putting everything together, we have

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}[D]=\sqrt{2/\pi}\int_0^\infty\frac{\mathrm{d}x}{\cosh(x)} &= 2\sqrt{2/\pi}\int_0^\infty\frac{\mathrm{e}^x\mathrm{d}x}{\mathrm{e}^{2x}+1}\\ &= 2\sqrt{2/\pi}\int_1^\infty\frac{\mathrm{d}y}{y^2+1}\\ &= 2\sqrt{\frac{2}{\pi}\frac{\pi}{4}}=\sqrt{\frac{\pi}{2}}.\end{aligned}$$

Exercise 2

Fix a probability space $(\Omega,\mathcal{F},\mathbb{P})$, and let B be a Brownian motion in $\mathbb{R}^d$ (with respect to its $\mathbb{P}$-completed natural filtration) for some integer $d\geq 2$. For any $x\in\mathbb{R}^d$, we let $B^x:=x+B$, and for any $x\in\mathbb{R}^d\setminus\{0\}$, and any $0<a<\|x\|<b$, we let

$$\tau_a:=\inf\{t\geq 0:\|B_t^x\|\leq a\},\ \tau_b:=\inf\{t\geq 0:\|B_t^x\|\geq b\}.$$

1) Assume $d\geq 3$ and show that $X_t^x:=\left\|B^x_{\tau_a\wedge t}\right\|^{2-d}$, $t\geq 0$, is a bounded $(\mathbb{F},\mathbb{P})$-martingale.

2) Assume that $d=2$, and show that $Y_t^x:=-\log\left(\left\|B^x_{\tau_a\wedge\tau_b\wedge t}\right\|\right)$, $t\geq 0$, is a bounded $(\mathbb{F},\mathbb{P})$-martingale.

3) Show that for any $x\in\mathbb{R}^d\setminus\{0\}$, $\mathbb{P}\left[B_t^x\neq 0,\ \forall t\geq 0\right]=1$.

4) Assume $d\geq 3$, and show that for any $x\in\mathbb{R}^d$, $\mathbb{P}\left[\lim_{t\to+\infty}\|B_t^x\|=+\infty\right]=1$.

1) **For any C^2 function $g:(0,+\infty)\longrightarrow\mathbb{R}$, we can define the radial function $f:\mathbb{R}^d\setminus\{0\}\longrightarrow\mathbb{R}$ by**

$$f(x):=g(\|x\|),\ x\in\mathbb{R}^d\setminus\{0\}.$$

Direct computations show that the Laplacian of f is given by

$$\Delta f(x)=g''(\|x\|)+\frac{d-1}{\|x\|}g'(\|x\|),\ x\in\mathbb{R}^d\setminus\{0\}.$$

Consider the map $g(r) := r^{2-d}$, which is C^2 on $(0,+\infty)$. We then have

$$\Delta f(x) = (2-d)(1-d)\|x\|^{-d} + (2-d)(d-1)\|x\|^{-d} = 0,\ x \in \mathbb{R}^d \setminus \{0\},$$

which means that f is harmonic on $\mathbb{R}^d \setminus \{0\}$. By applying Itô's formula, we deduce

$$X_t^x = \left\|B^x_{\tau_a\wedge t}\right\|^{2-d} = f\big(B^x_{\tau_a\wedge t}\big) = f(x) + \int_0^{\tau_a\wedge t} \partial_x f(B_s^x)\mathrm{d}B_s,$$

from which the $(\mathbb{F},\mathbb{P})$–local martingale property is immediate. In addition, since $d\geq 3$, we have that

$$0 \leq X_t^x \leq a^{2-d},$$

which provides the required boundedness property, and ensures that we have an $(\mathbb{F},\mathbb{P})$-martingale.

2) **It is the exact same argument as in** 1) **except with $g(r) := -\log(r)$.**

3) **For the cases $d \geq 3$ and $d=2$, we let g and f be as in** 1) **and** 2) **respectively. Consider then the martingale $M_t^x := f(B^x_{t\wedge\tau_a\wedge\tau_b})$, $t\geq 0$ (and notice that this coincides with X^x when $d\geq 3$), and recall that $\tau_a\wedge\tau_b$ is finite $\mathbb{P}$–a.s. because τ_b is. Since M^x is bounded, it is $\mathbb{P}$–uniformly integrable and**

$$g(\|x\|) = M_0^x = \mathbb{E}^{\mathbb{P}}\left[\lim_{t\to+\infty} M_t\right] = g(a)\mathbb{P}[\tau_a \leq \tau_b] + g(b)\mathbb{P}[\tau_b < \tau_a],$$

which leads to

$$\mathbb{P}[\tau_a \leq \tau_b] = \frac{g(\|x\|) - g(b)\mathbb{P}[\tau_b<\tau_a]}{g(a)}.$$

Since in both cases, g goes to $+\infty$ at 0, we obtain by letting a go to 0 and using dominated convergence

$$\mathbb{P}[\tau_0 \leq \tau_b] = 0,\ b > \|x\|.$$

Taking the limit again as b goes to $+\infty$, we now get

$$\mathbb{P}\left[\tau_0 < \lim_{b\to+\infty}\tau_b\right] = 0.$$

Since B^x is $(\mathbb{F},\mathbb{P})$–locally bounded (it is continuous), we must have $\lim_{b\to+\infty}\tau_b = +\infty$, so that $\tau_0 = +\infty$, $\mathbb{P}$–a.s. which is the desired result.

4) **By translation invariance of Brownian motion, we can assume without loss of generality that $x\neq 0$. For $d\geq 3$, define**

$$M_t^x := g\big(\|B_t^x\|\big) = \|B_t^x\|^{2-d}.$$

This is $\mathbb{P}$–a.s. well defined for all $t\geq 0$, since $\tau_0 = +\infty$, $\mathbb{P}$–a.s. by 3). As in 1), M^x is an $(\mathbb{F},\mathbb{P})$–local martingale. In this case, while M^x is not bounded, it is a non-negative $(\mathbb{F},\mathbb{P})$–local martingale, hence an $(\mathbb{F},\mathbb{P})$–super-martingale. Since M^x is a non-negative $(\mathbb{F},\mathbb{P})$–super-martingale, it also follows by the super-martingale convergence theorem that $M_t^x \longrightarrow_{t\to+\infty} M_\infty^x$, $\mathbb{P}$–a.s. for some $\mathcal{F}_{\infty-}$-measurable random variable M_∞^x. Noting that

$$\limsup_{t\to+\infty}\|B_t^x\| = +\infty,\ \mathbb{P}\text{–a.s.},$$

so that

$$M_\infty^x = \lim_{t\to+\infty} M_t^x = \liminf_{t\to+\infty}\|B_t^x\|^{2-d} = 0,\ \mathbb{P}\text{–a.s.},$$

and thus as desired

$$\|B_t^x\| = (M_t^x)^{1/(2-d)} \underset{t\to+\infty}{\longrightarrow} +\infty,\ \mathbb{P}\text{–a.s.}$$

Exercise 3

Let B be a Brownian motion in $\mathbb{R}^3$, $0\neq x\in\mathbb{R}^3$ and define the process $M=(M_t)_{t\geq 0}$ by

$$M_t = \frac{1}{\|x+B_t\|}.$$

This is well defined as a 3-dimensional Brownian motion does not hit points, as seen in the previous exercise.

1) Show that M is a continuous local martingale. Moreover, show that M is bounded in $\mathbb{L}^2(\mathbb{R},\mathcal{F},\mathbb{P})$, that is

$$\sup_{t\geq 0}\mathbb{E}^{\mathbb{P}}[|M_t|^2]<+\infty.$$

2) Show that M is a *strict local martingale*, i.e., M is not a martingale.

Hint: Show that $\mathbb{E}^{\mathbb{P}}[M_t]\longrightarrow 0$ as $t\to+\infty$. To this end, similarly to 1), compute $\mathbb{E}^{\mathbb{P}}[M_t]$ and use the reverse triangle inequality as a first estimate. Then compute the resulting integral using spherical coordinates.

1) **Since the 3-dimensional Brownian motion $B=(B^1,B^2,B^3)^\top$ takes values in the open set $D:=\mathbb{R}^d\setminus\{-x\}$, $\mathbb{P}$–a.s., we can apply Itô's formula to $M_t=f(B_t)$ with $f:D\longrightarrow(0,+\infty)$ given by $f(y):=\frac{1}{\|x+y\|}$.**

For $i\in\{1,2,3\}$, we have

$$\frac{\partial f}{\partial y^i}(y)=-\frac{x^i+y^i}{\|x+y\|^3},\ \frac{\partial^2 f}{(\partial y^i)^2}(y)=\frac{-\|x+y\|^2+3(x^i+y^i)^2}{\|x+y\|^5}.$$

It follows that $\Delta f=\frac{\partial^2 f}{(\partial y^1)^2}+\frac{\partial^2 f}{(\partial y^2)^2}+\frac{\partial^2 f}{(\partial y^3)^2}=0$ on D. Hence, Itô's formula yields

$$M_t=M_0+\int_0^t\nabla f(B_s)\cdot\mathrm{d}B_s+\frac{1}{2}\int_0^t\Delta f(B_s)\mathrm{d}s=\frac{1}{\|x\|}-\sum_{i=1}^3\int_0^t\frac{x^i+B_s^i}{\|x+B_s\|^3}\mathrm{d}B_s^i.$$

Thus, M is a continuous $(\mathbb{F},\mathbb{P})$–local martingale.

Let us show the second part. For $t>0$, using the distribution of the 3-dimensional Brownian motion, we obtain that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[|M_t|^2\mathbf{1}_{\{|M_t|\geq\frac{2}{\|x\|}\}}\big]&=(2\pi t)^{-\frac{3}{2}}\int_{\|x+y\|\leq\frac{\|x\|}{2}}\frac{1}{\|x+y\|^2}\exp\bigg(-\frac{\|y\|^2}{2t}\bigg)\mathrm{d}y\\&=(2\pi t)^{-\frac{3}{2}}\int_{\|y\|\leq\frac{\|x\|}{2}}\frac{1}{\|y\|^2}\exp\bigg(-\frac{\|y-x\|^2}{2t}\bigg)\mathrm{d}y\\&\leq(2\pi t)^{-\frac{3}{2}}\int_{\|y\|\leq\frac{\|x\|}{2}}\frac{1}{\|y\|^2}\exp\bigg(-\frac{(\|x\|-\|y\|)^2}{2t}\bigg)\mathrm{d}y\\&\leq(2\pi t)^{-\frac{3}{2}}\exp\bigg(-\frac{\|x\|^2}{8t}\bigg)\int_{\|y\|\leq\frac{\|x\|}{2}}\frac{1}{\|y\|^2}\mathrm{d}y\\&=(2\pi t)^{-\frac{3}{2}}\exp\bigg(-\frac{\|x\|^2}{8t}\bigg)\int_0^{\frac{\|x\|}{2}}\int_0^{2\pi}\int_0^{\pi}\frac{1}{r^2}r^2\sin(\theta)\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}r\\&\leq C(2\pi t)^{-\frac{3}{2}}\exp\bigg(-\frac{\|x\|^2}{8t}\bigg),\end{aligned}$$

where C is a finite positive constant.

Now, the function $t\longmapsto(2\pi t)^{-\frac{3}{2}}\exp\big(-\frac{\|x\|^2}{8t}\big)$ is continuous on $(0,+\infty)$ and converges to 0 as $t\to 0$ and $t\to\infty$, hence it is bounded on $(0,\infty)$. Therefore, we conclude that

$$\sup_{t\geq 0}\mathbb{E}^{\mathbb{P}}[|M_t|^2]\leq\frac{4}{|x|^2}+\sup_{t\geq 0}\mathbb{E}^{\mathbb{P}}\big[|M_t|^2\mathbf{1}_{\{|M_t|\geq\frac{2}{\|x\|}\}}\big]<+\infty.$$

It follows that M is bounded in $\mathbb{L}^2(\mathbb{R},\mathcal{F},\mathbb{P})$.

2) **For $t>0$, using spherical coordinates,**

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[M_t] &= (2\pi t)^{-3/2}\int_{\mathbb{R}^3}\frac{1}{\|x+y\|}\exp\bigg(-\frac{\|y\|^2}{2t}\bigg)\mathrm{d}y\\
&= (2\pi t)^{-3/2}\int_{\mathbb{R}^3}\frac{1}{\|y\|}\exp\bigg(-\frac{\|y-x\|^2}{2t}\bigg)\mathrm{d}y\\
&\leq (2\pi t)^{-3/2}\int_{\mathbb{R}^3}\frac{1}{\|y\|}\exp\bigg(-\frac{(\|y\|-\|x\|)^2}{2t}\bigg)\mathrm{d}y\\
&= (2\pi t)^{-3/2}\int_0^\infty\int_0^{2\pi}\int_0^\pi\frac{1}{r}\exp\bigg(-\frac{(r-\|x\|)^2}{2t}\bigg)r^2\sin(\theta)\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}r\\
&= 4\pi(2\pi t)^{-3/2}\int_0^\infty r\exp\bigg(-\frac{(r-\|x\|)^2}{2t}\bigg)\mathrm{d}r\\
&= 4\pi(2\pi t)^{-3/2}\int_{-\|x\|}^\infty(r+\|x\|)\exp\bigg(-\frac{r^2}{2t}\bigg)\mathrm{d}r\\
&= 4\pi(2\pi t)^{-3/2}\bigg(\int_{-\|x\|}^\infty r\exp\bigg(-\frac{r^2}{2t}\bigg)\mathrm{d}r+\|x\|\int_{-\|x\|}^\infty\exp\bigg(-\frac{r^2}{2t}\bigg)\mathrm{d}r\bigg)\\
&\leq 4\pi(2\pi t)^{-3/2}\bigg(\bigg[-t\exp\bigg(-\frac{r^2}{2t}\bigg)\bigg]_{-\|x\|}^\infty+\|x\|\sqrt{2\pi t}\bigg)\\
&= 4\pi(2\pi t)^{-3/2}\bigg(t\exp\bigg(-\frac{\|x\|^2}{2t}\bigg)+\|x\|\sqrt{2\pi t}\bigg)=O\big(t^{-\frac{1}{2}}\big),\ (t\longrightarrow+\infty).
\end{aligned}$$

Hence, $\mathbb{E}^{\mathbb{P}}[M_t]\longrightarrow 0$ as $t\longrightarrow+\infty$. Since $\mathbb{E}^{\mathbb{P}}[M_0]=\frac{1}{\|x\|}>0$, M cannot be an $(\mathbb{F},\mathbb{P})$-martingale.

Exercise 4

Let B be a Brownian motion. For all $y\in\mathbb{R}_+^\star$, we define

$$T_y:=\inf\big\{t\geq 0: B_t\geq y\big\}.$$

Fix $a>0$ and $b>0$ and define

$$T_{a,b}:=T_{-a}\wedge T_b.$$

1) Justify that $T_{a,b}$ is an $\mathbb{F}^{B,\mathbb{P}}$–stopping time.
2) Fix $\theta\in\mathbb{R}$ and define $X_t^{\theta,a}$ by
$$X_t^{\theta,a}:=\sinh(\theta(B_t+a))\exp\bigg(-\frac{\theta^2}{2}t\bigg).$$
Show that $X^{\theta,a}$ is an $(\mathbb{F}^{B,\mathbb{P}},\mathbb{P})$-martingale.
3) Deduce that
$$\mathbb{E}^{\mathbb{P}}\bigg[\exp\bigg(-\frac{\theta^2}{2}T_b\bigg)\mathbf{1}_{\{T_b<T_{-a}\}}\bigg]=\frac{\sinh(\theta a)}{\sinh(\theta(a+b))},$$
and then that
$$\mathbb{E}^{\mathbb{P}}\bigg[\exp\bigg(-\frac{\theta^2}{2}T_{-a}\bigg)\mathbf{1}_{\{T_b>T_{-a}\}}\bigg]=\frac{\sinh(\theta b)}{\sinh(\theta(a+b))},$$
and finally that
$$\mathbb{E}\bigg[\exp\bigg(-\frac{\theta^2}{2}T_{a,b}\bigg)\bigg]=\frac{\cosh\big(\frac{\theta(a-b)}{2}\big)}{\cosh\big(\frac{\theta(a+b)}{2}\big)}.$$

4) Deduce

$$\mathbb{P}\big[T_b < T_{-a}\big] = \frac{a}{a+b},\ \mathbb{P}\big[T_b > T_{-a}\big] = \frac{b}{a+b},$$

and then that the random variable $\sup_{0\leq t\leq T_{-1}} B_t$ has the same law as $(1-U)/U$ where U is uniform on $[0,1]$.

1) **It is the minimum between two stopping times, and thus a stopping time.**

2) **The (local) martingale property is direct from, for instance, Itô's formula. Since B has exponential moments of any order, one can check that it is, in fact, a true martingale.**

3) **The martingale property and the optional sampling theorem ensure that, with $T^n_{a,b} := T_{a,b}\wedge n$, for any $n\in\mathbb{N}$,**

$$\mathbb{E}^{\mathbb{P}}\bigg[\sinh\big(\theta\big(B_{T^n_{a,b}}+a\big)\big)\exp\bigg(-\frac{\theta^2}{2}T^n_{a,b}\bigg)\bigg] = X_0^{\theta,a} = \sinh(\theta a).$$

Now we also have

$$\sinh\big(\theta\big(B_{T^n_{a,b}}+a\big)\big)\exp\bigg(-\frac{\theta^2}{2}T^n_{a,b}\bigg) = \mathbf{1}_{\{T_{a,b}\geq n\}}\sinh(\theta(B_n+a))\mathrm{e}^{-\frac{\theta^2}{2}n}+\mathbf{1}_{\{T_{a,b}<n\}}\mathbf{1}_{\{T_b<T_{-a}\}}\sinh(\theta(b+a))\exp\bigg(-\frac{\theta^2}{2}T_b\bigg),$$

and the left-hand side converges $\mathbb{P}$–a.s. to $\mathbf{1}_{\{T_b<T_{-a}\}}\sinh(\theta(b+a))\exp\big(-\frac{\theta^2}{2}T_b\big)$ as n goes to $+\infty$, since by, for instance, the law of iterated logarithm for Brownian motion, $T_{a,b}<+\infty$, $\mathbb{P}$–a.s. Now we also have that

$$\bigg|\sinh\big(\theta\big(B_{T^n_{a,b}}+a\big)\big)\exp\bigg(-\frac{\theta^2}{2}T^n_{a,b}\bigg)\bigg| \leq \sinh(\theta(b+a)),$$

so that by dominated convergence, we get

$$\mathbb{E}^{\mathbb{P}}\bigg[\mathbf{1}_{\{T_b<T_{-a}\}}\sinh(\theta(b+a))\exp\bigg(-\frac{\theta^2}{2}T_b\bigg)\bigg] = \sinh(\theta a),$$

which is the first stated equality.

For the second one, by symmetry for Brownian motion, we can apply the previous result to $-B$, and use that $\sinh(-x) = -\sinh(x)$. The last one follows immediately by adding the two previously obtained equalities and standard formulas for hyperbolic functions.

4) **For the first part, it suffices to use again dominated convergence and to let θ go to 0 in the previous first two equalities. Finally, we have**

$$\mathbb{P}\bigg[\sup_{0\leq t\leq T_{-1}} B_t < x\bigg] = \mathbb{P}[T_x > T_{-1}] = \frac{x}{x+1},\ \mathbb{P}[(1-U)/U<x] = \mathbb{P}[U>1/(x+1)] = 1-\frac{1}{x+1} = \frac{x}{x+1},$$

which is the desired result.